Distribution Sécurisée du Temps et Systèmes Spatiaux

Grenoble, 13-15 novembre 2024

Concepts et Outils de la Métrologie Temps-Fréquence

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FIRST-TF

Outline

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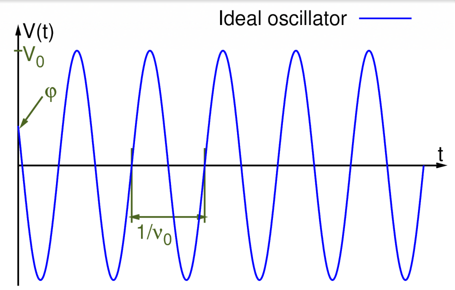
Introduction

Notations in the time domain



$$V(t) = V_0 \sin\left[2\pi\nu_0 t + \varphi\right]$$

where $\varphi(t)$ is the phase "*noise*"

- **Time error (or phase time) $x(t)$:**

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with $\quad x(t) = \dfrac{\varphi(t)}{2\pi\nu_0} \quad [s]$

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- t_{watch} = 10 h 10 min 37 s
- t_{ref} = 10 h 11 min 16 s

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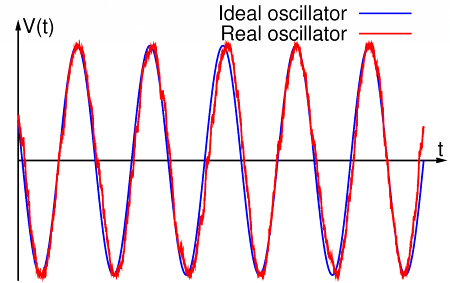

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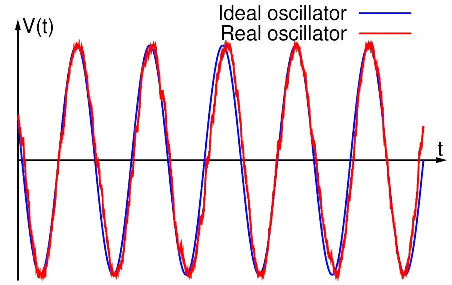

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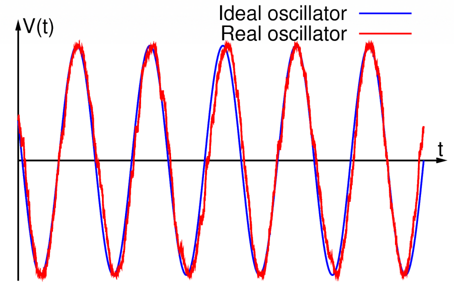

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- **Frequency deviation $y(t)$:**

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Frequency noise *vs* Phase noise

Phase and frequency noise: 2 representations of 1 phenomenon

$$\left.\begin{aligned} x(t) &= \frac{\varphi(t)}{2\pi\nu_0} \\ y(t) &= \frac{1}{2\pi\nu_0}\frac{d\varphi(t)}{dt} \end{aligned}\right\} \Rightarrow \quad y(t) = \frac{dx(t)}{dt}$$

A fundamental difference:

- $\varphi(t)$ and $x(t)$ are instantaneous
- $\Delta\nu(t)$ and $y(t)$ have to be averaged

Example of a Π-counter:

$$\bar{y}_k = \frac{1}{\tau}\int_{t_k}^{t_k+\tau} y(t)dt = \frac{x(t_k+\tau) - x(t_k)}{\tau}$$

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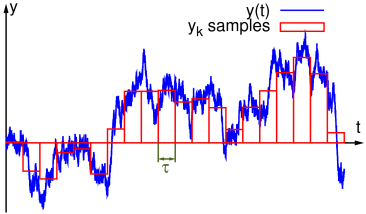

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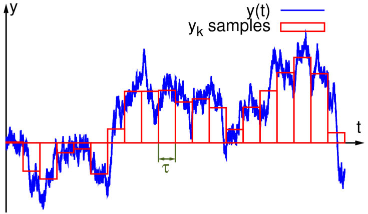

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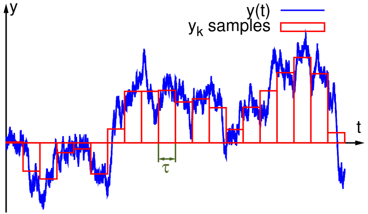
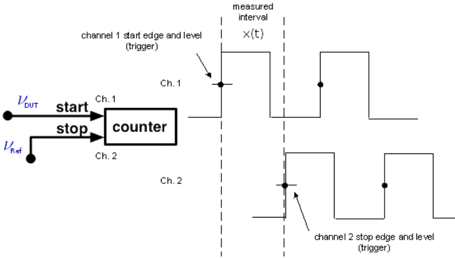


Phase-time measurements: the time interval counter

Measure the time between

- the rising edge of the DUT signal
- the rising edge of the reference signal



Conversion to frequency deviation: $\bar{y}_k = \frac{x(t_k+\tau)-x(t_k)}{\tau}$

- **Available sampling rate:** several MSample/s
- **Needs:** $\tau \sim 1$ s!

$\Rightarrow$ Possible to '**shape**' the $\bar{y}_k$ by weighted average:

Π counter   Λ counter   Ω counter

best rejection of the white phase noise ↗

Other measurements and methods

Frequency counter, phasemeter, beat-note method, frequency comb (optical domain)...

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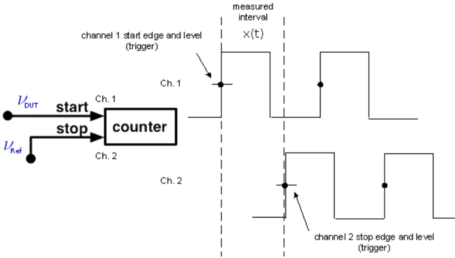

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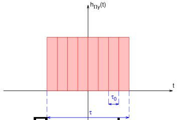

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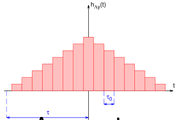

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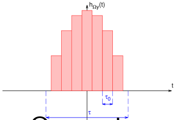

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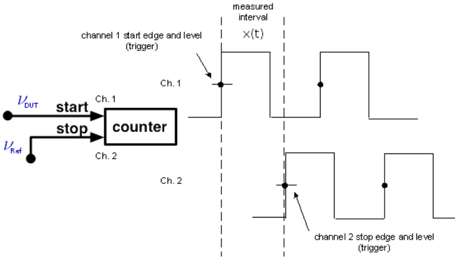

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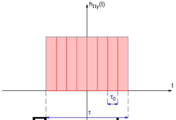

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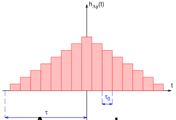

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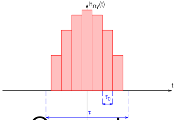

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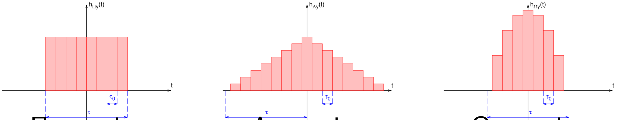

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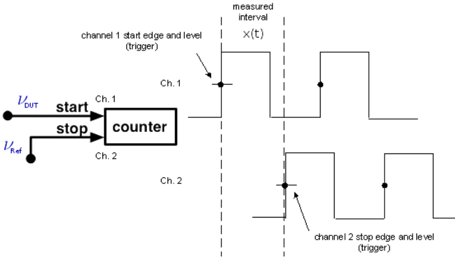

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Power Spectral Densities (PSD)

- Fourier Transform (finite energy):

$$\Phi(f) = \int_{-\infty}^{+\infty} \varphi(t) e^{-j2\pi ft} \mathrm{d}t \qquad [s]$$

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Short-term stability: the phase noise

The one-sided PSD

$\varphi(t)$ is real

$$\mathcal{I}\left[\varphi(t)\right] = 0 \quad \Leftrightarrow \quad S_\varphi(f) = S_\varphi(-f)$$

Definition of the "one-sided" phase Power Spectral density $S_\varphi(f)^{1S}$:

$$\begin{cases} S_\varphi(f)^{1S} = 2S_\varphi(f)^{2S} & \text{if } f \geq 0 \\ S_\varphi(f)^{1S} = 0 & \text{if } f < 0 \end{cases}$$

Units on log-log plots

$10\log_{10}\left[S_\varphi(f)\right]$ is expressed in **dB/Hz**

$L(f)$, a survival from the past

In many data shets, one finds

$L(f) = 10\log_{10}\left[\frac{1}{2}S_\varphi(f)\right]$ is expressed in **dBc/Hz**

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Relationships between PSD

Time error PSD: $S_x(f)$

- $x(t) = \dfrac{\varphi(t)}{2\pi\nu_0}$ $\Rightarrow$ $S_x(f) = \dfrac{1}{4\pi^2\nu_0^2} S_\varphi(f)$
- Dimension: $[s^3] \equiv [Hz^{-3}]$

Frequency deviation PSD: $S_y(f)$

- $y(t) = \dfrac{1}{2\pi\nu_0}\dfrac{d\varphi(t)}{dt}$ $\Rightarrow$ $S_y(f) = \dfrac{f^2}{\nu_0^2} S_\varphi(f)$
- $y(t) = \dfrac{dx(t)}{dt}$ $\Rightarrow$ $S_y(f) = 4\pi^2 f^2 S_x(f)$
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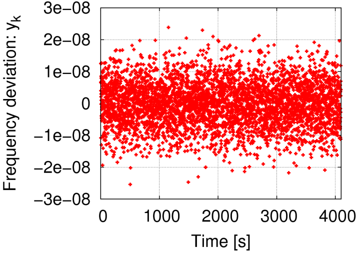
Noise model

The power law noise model

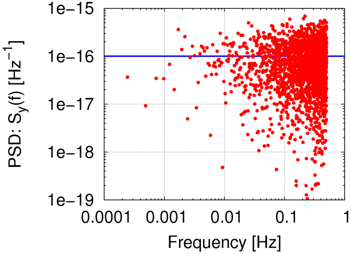
$$S_y(f) = \sum_{\alpha=-2}^{+2} h_\alpha f^\alpha \qquad \alpha \text{ integer}$$

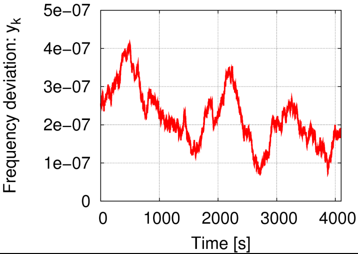
$S_y(f)$	$S_\varphi(f)$	Noise type	Origin
$h_{-2}f^{-2}$	$b_{-4}f^{-4}$	Random Walk Freq. Mod.	Environment
$h_{-1}f^{-1}$	$b_{-3}f^{-3}$	Flicker F.M.	Resonator
h_0	$b_{-2}f^{-2}$	White F.M.	Thermal noise
$h_1 f$	$b_{-1}f^{-1}$	Flicker Phase Mod.	Electronic noise
$h_2 f^2$	b_0	White P.M.	External white noise

White FM *vs* Random Walk FM

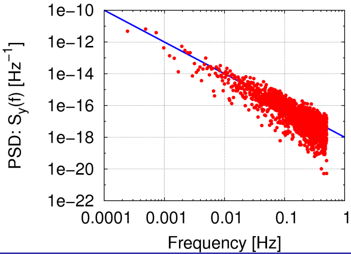


White FM

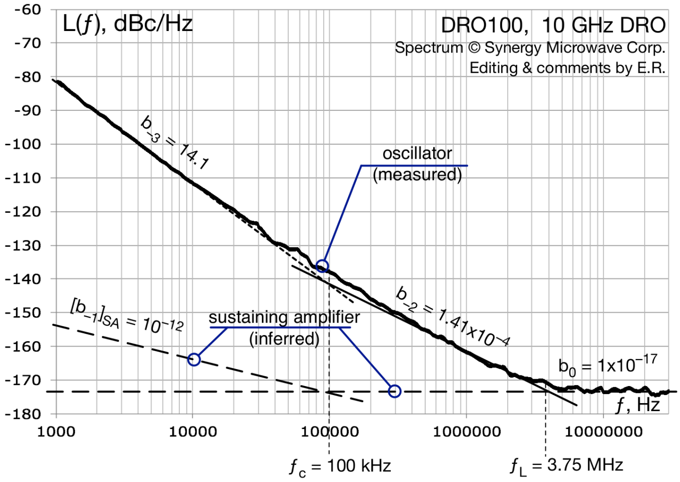




Random Walk FM



Phase noise measurement



Courtesy of Ulrich L. Rohde, Synergy Microwave Corporation

Outline

A statistical estimator

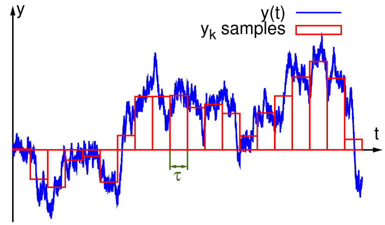
as well as a spectral analysis tool

- Definition of the true variance:

$$I^2(\tau) = \left\langle \left(\bar{y}_k - \langle \bar{y}_k \rangle\right)^2 \right\rangle$$

- Estimation of the true variance:

$$\sigma^2(N,\tau) = \frac{1}{N-1} \sum_{i=1}^{N} \left(\bar{y}_i - \frac{1}{N} \sum_{j=1}^{N} \bar{y}_j \right)^2$$



- The Allan variance (2-sample variance):

$$\sigma_y^2(\tau) = \sigma^2(2,\tau) = \sum_{i=1}^{2} \left(\bar{y}_i - \frac{1}{2} \sum_{j=1}^{2} \bar{y}_j \right)^2$$

$\langle\rangle$ stands for:

- ensemble average
- time average

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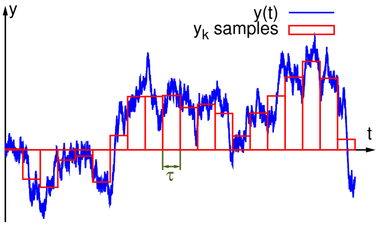

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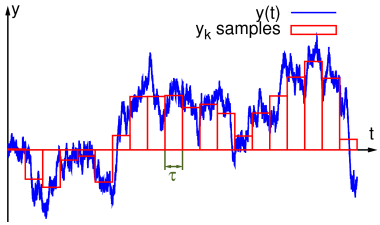

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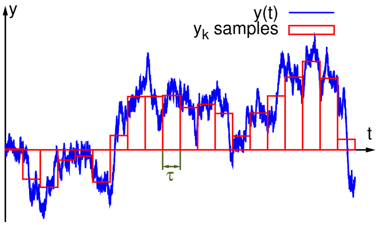

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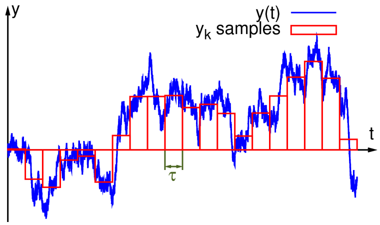

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A spectral analysis tool

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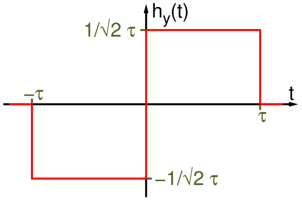
Convolution in the time domain…

$$\sigma_y^2(\tau) = \left\langle \left[\int_{-\infty}^{+\infty} y(t) h_y(t_k - t) \mathrm{d}t \right]^2 \right\rangle$$

$$\text{with} \begin{cases} h_y(t) = \dfrac{-1}{\sqrt{2\tau}} & \text{if} \quad -\tau \geq t < 0 \\ h_y(t) = \dfrac{+1}{\sqrt{2\tau}} & \text{if} \quad 0 \geq t < \tau \\ h_y(t) = 0 & \text{else} \end{cases}$$



…filtering in the frequency domain

$$\sigma_y^2(\tau) = \int_0^{\infty} S_y(f) \left| H_y(f) \right|^2 \mathrm{d}f$$

$$\text{with } |H_y(f)|^2 = |\mathrm{FT}[h_y(t)]|^2 = 2 \frac{\sin^4(\pi\tau f)}{(\pi\tau f)^2}$$

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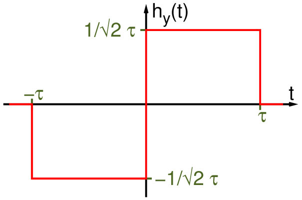

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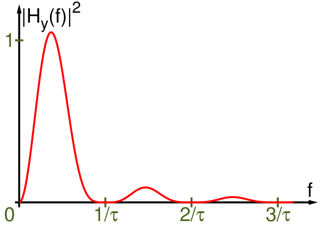
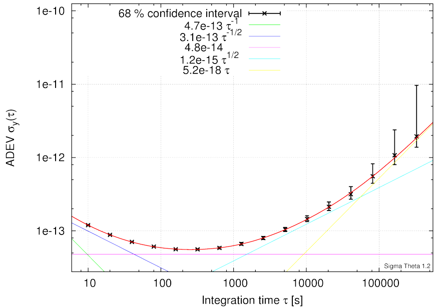


Link between noise levels and variance responses

$$\sigma_y^2(\tau) = 2\int_0^{+\infty} h_\alpha f^\alpha \frac{\sin^4(\pi\tau f)}{(\pi\tau f)^2} df$$

f_h is the high cut-off frequency



$S_y(f)$	$h_{-2}f^{-2}$	$h_{-1}f^{-1}$	$h_0 f^0$	$h_{+1}f^{+1}$	$h_{+2}f^{+2}$
$\sigma_y^2(\tau)$	$\frac{2\pi^2 h_{-2}\tau}{3}$	$2\ln(2)h_{-1}$	$\frac{h_0}{2\tau}$	$\frac{[1.04 + 3\ln(2\pi f_h \tau)]\, h_{+1}}{4\pi^2\tau^2}$	$\frac{3h_{+2}f_h}{4\pi^2\tau^2}$

and $\sigma_y^2(\tau) = \frac{1}{2}D_1^2\tau^2$ for a linear frequency drift: $y(t) = D_1 t$

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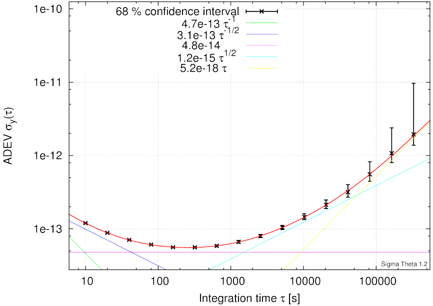

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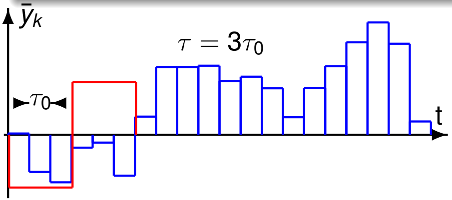
Allan variance with or without overlapping

Allan variance with overlapping

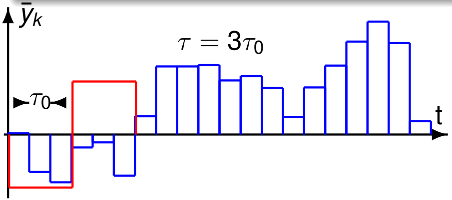


τ_0-steps moving average

Benefits and drawbacks :

- lower dispersion
- more correlated estimates

Allan variance without overlapping



Shifted by τ-steps :

$\tau = 3\tau_0 \Leftrightarrow \bar{Y}_1 = (\bar{y}_1 + \bar{y}_2 + \bar{y}_3)/3$

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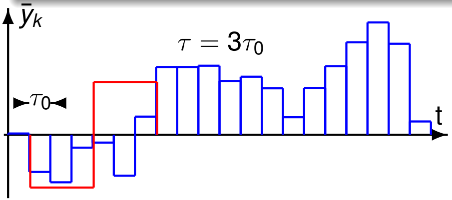

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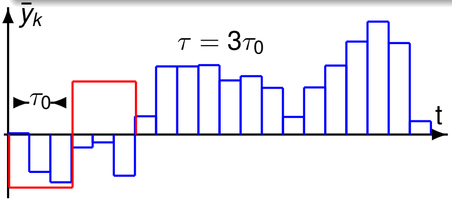

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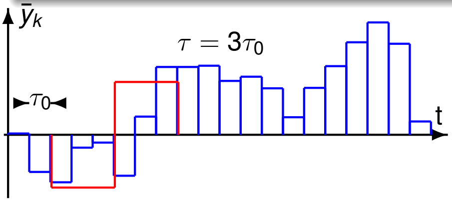

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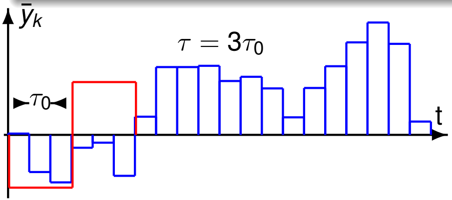

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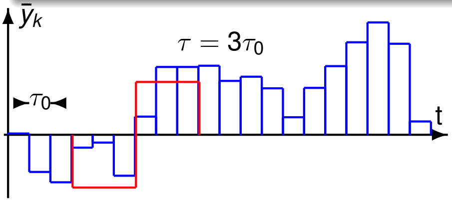

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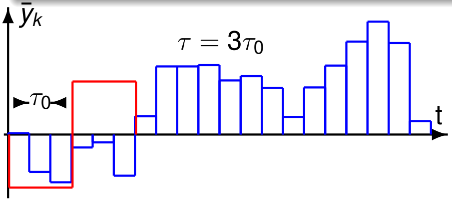

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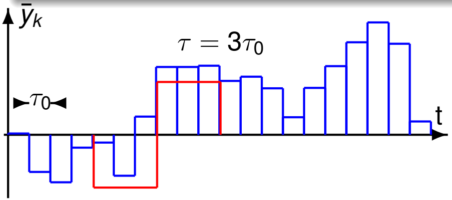

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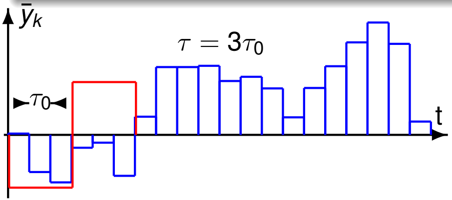

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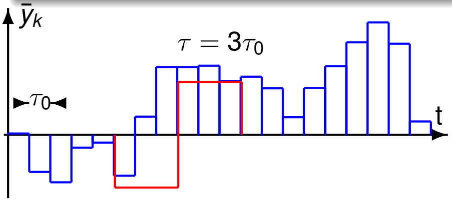

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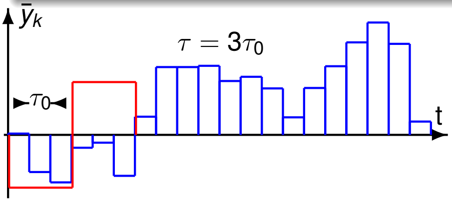

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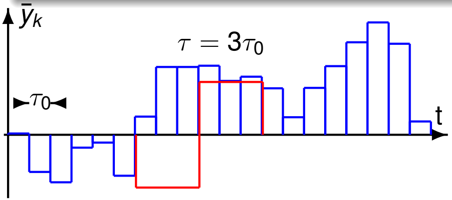

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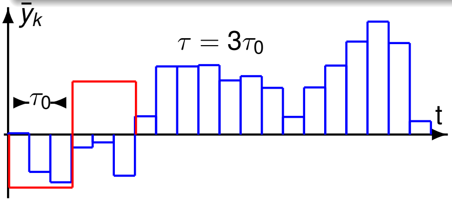

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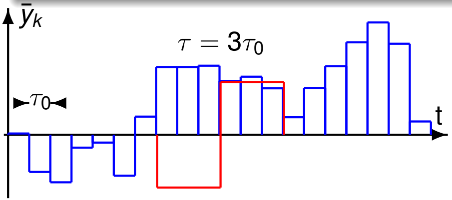

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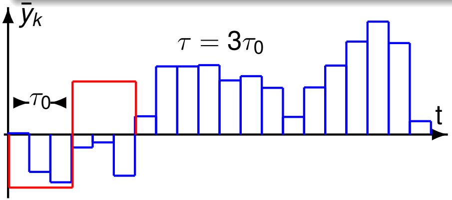

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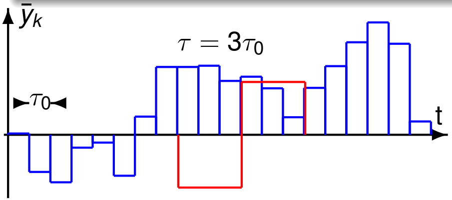

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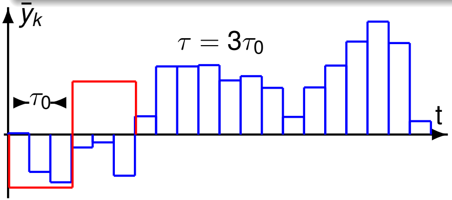

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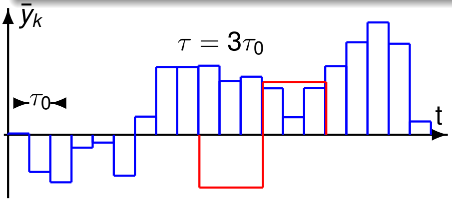

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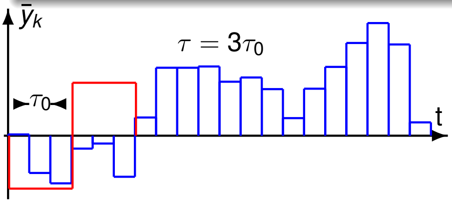

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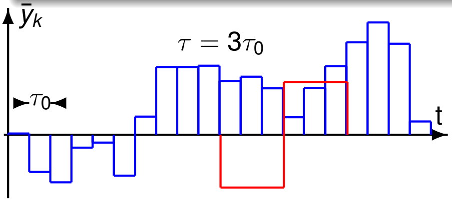

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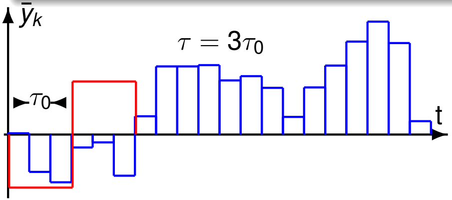

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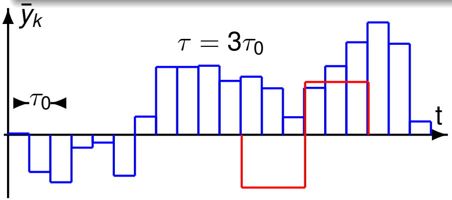

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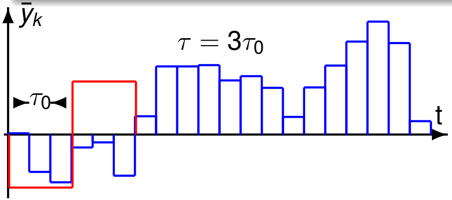

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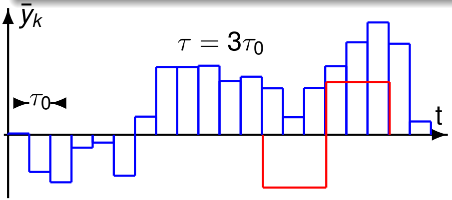

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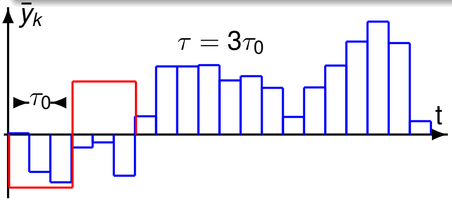

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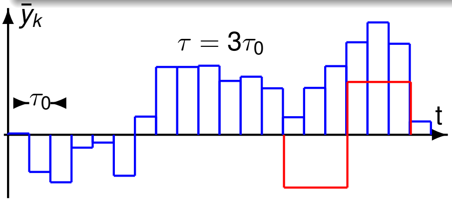

τ_0-steps moving average

Benefits and drawbacks :

- lower dispersion
- more correlated estimates

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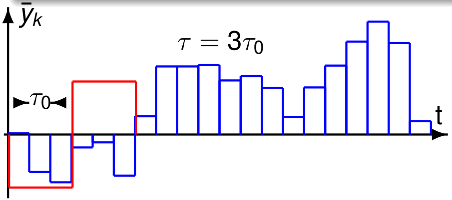

Shifted by τ-steps :

$\tau = 3\tau_0 \Leftrightarrow \bar{Y}_1 = (\bar{y}_1 + \bar{y}_2 + \bar{y}_3)/3$

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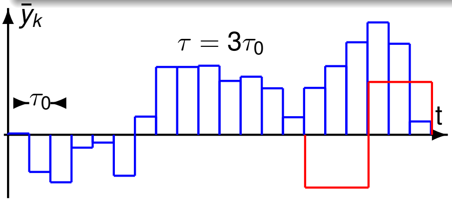

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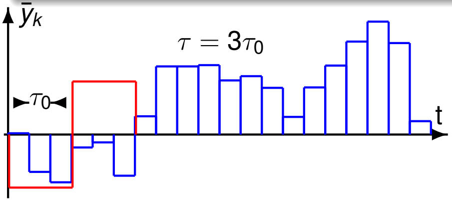

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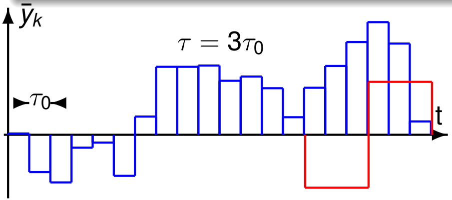

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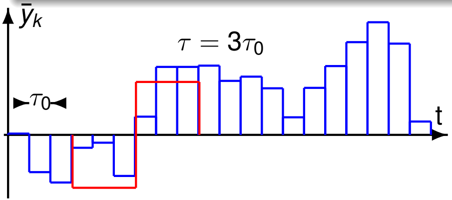

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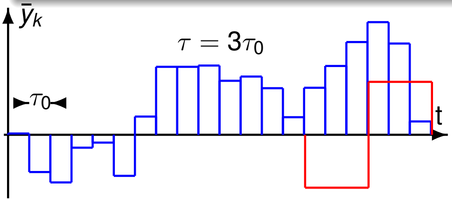

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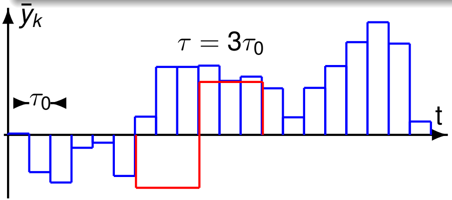

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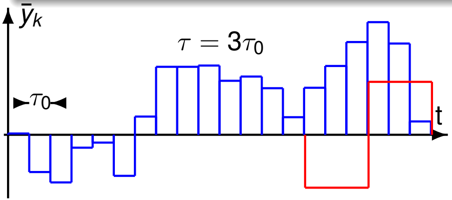

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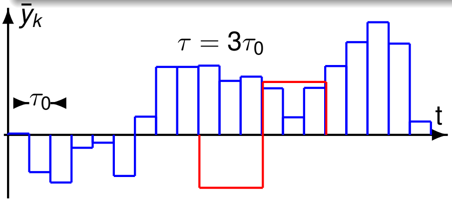

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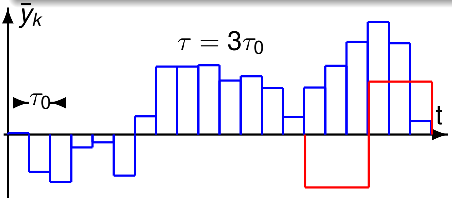

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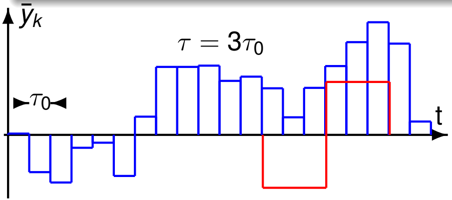

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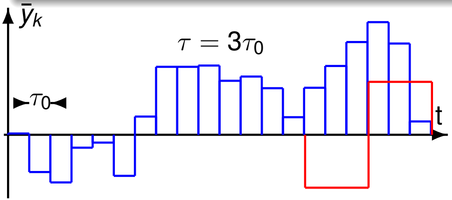

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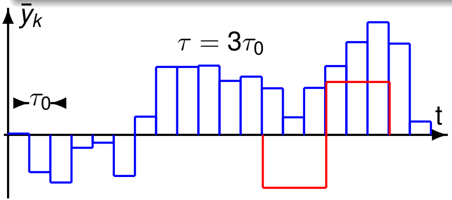

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Allan variance versus Allan deviation

$$\mathrm{ADEV}(\tau) = \sigma_y(\tau) = \sqrt{\sigma_y^2(\tau)}$$

Physical meaning

- $\sigma_y(\tau) = \dfrac{\Delta t}{\tau}$

  Ex.: Cs clock $\quad \sigma_y(\tau = 1\mathrm{day}) = 10^{-14} \quad \Rightarrow \quad \Delta t \approx 10^{-14} \cdot 10^5 = 10^{-9} = 1$ ns over 1 day

- $\sigma_y(\tau) = \dfrac{\Delta f}{\nu_0}$ (during τ)

  Ex.: H-Maser @ 100 MHz $\sigma_y(\tau = 1\mathrm{h}) = 10^{-14} \Rightarrow \Delta f \approx 10^{-14} \cdot 10^8 = 10^{-6} = 1\mu$Hz over 1 h

Benefits and drawbacks

- Easy to interpret
- Biased

Never fit the Allan deviation curves, always use the Allan variance!

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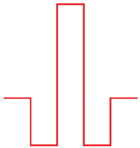
The most widely used variances

- **The Hadamard variance** (Picinbono):

$$\sigma_H^2(\tau) = \frac{1}{6}\left\langle \left(-\bar{y}_1 + 2\bar{y}_2 - \bar{y}_3\right)^2 \right\rangle .$$

$$|H_H(f)|^2 = \frac{8}{3}\frac{\sin^6(\pi\tau f)}{(\pi\tau f)^2}.$$



- **The modified Allan variance** (MVAR):

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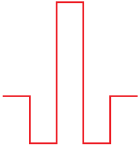

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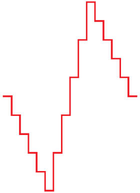

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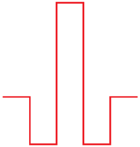

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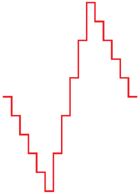

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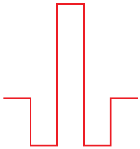

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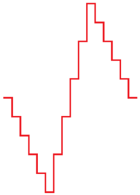

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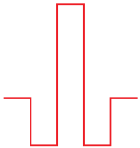

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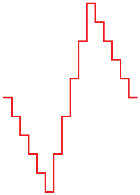

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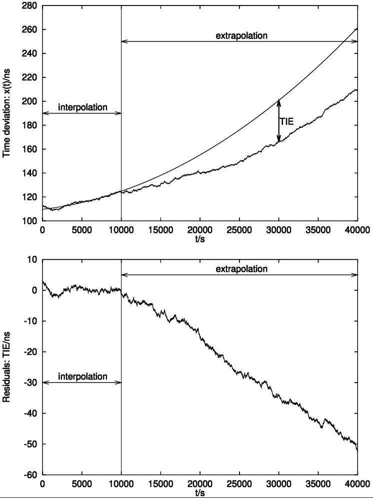
Responses of the variances

$S_y(f)$ (s $\equiv$ Hz^{-1})	$\sigma_H^2(\tau)$	Mod$\sigma_y^2(\tau)$	$\sigma_x^2(\tau)$
$h_{-4}f^{-4}$	$\frac{44\pi^4 h_{-4}\tau^3}{60}$	-	-
$h_{-3}f^{-3}$	$\frac{[27\ln(2)-32\ln(3)]\pi^2 h_{-3}\tau^2}{6}$	-	-
$h_{-2}f^{-2}$	$\frac{\pi^2 h_{-2}\tau}{3}$	$\frac{11\pi^2 h_{-2}\tau}{20}$	$\frac{11\pi^2 h_{-2}\tau^3}{60}$
$h_{-1}f^{-1}$	$\frac{[8\ln(2)-3\ln(3)]h_{-1}}{2}$	$\frac{[27\ln(3)-32\ln(2)]h_{-1}}{8}$	$\frac{[27\ln(3)-32\ln(2)]h_{-1}}{24\tau^2}$
$h_0 f^0$	$\frac{h_0}{2\tau}$	$\frac{h_0}{4\tau}$	$\frac{h_0\tau}{12}$
$h_{+1}f^{+1}$	$\frac{5[0{,}964+\ln(\pi\tau f_h)]h_{+1}}{6\pi^2\tau^2}$	$\frac{[24\ln(2)-9\ln(3)]h_{+1}}{8\pi^2\tau^2}$	$\frac{[8\ln(2)-3\ln(3)]h_{+1}}{8\pi^2}$
$h_{+2}f^{+2}$	$\frac{5h_{+2}f_h}{6\pi^2\tau^2}$	$\frac{3h_{+2}}{8\pi^2\tau^3}$	$\frac{h_{+2}}{8\pi^2\tau}$

Time Interval Error



TIE

The Time Interval Error (TIE) is the difference between the extrapolation of the clock model and the clock state at a given time.

MTIE

The Maximum Time Interval Error (MTIE) can be computed from **TDev** and the type of noise.

F. Vernotte, J. Delporte, M. Brunet, and T. Tournier. Uncertainties of drift coefficients and extrapolation errors: Application to clock error prediction. Metrologia, 38(4):325–342, 2001.

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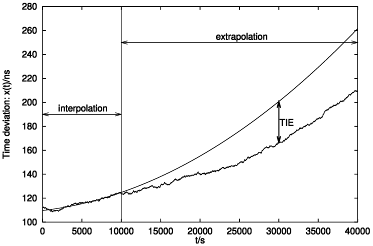

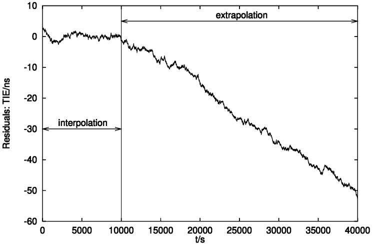

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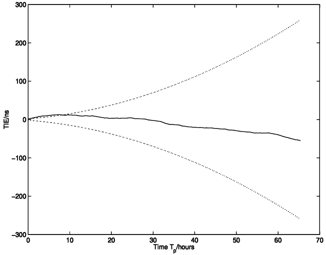
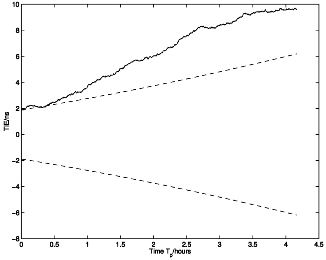
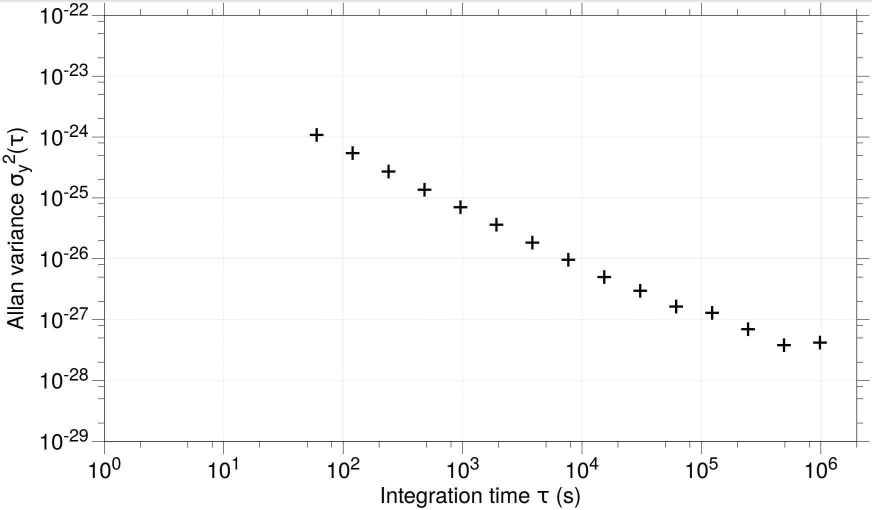




Basic interpretation of an Allan variance curve (I)



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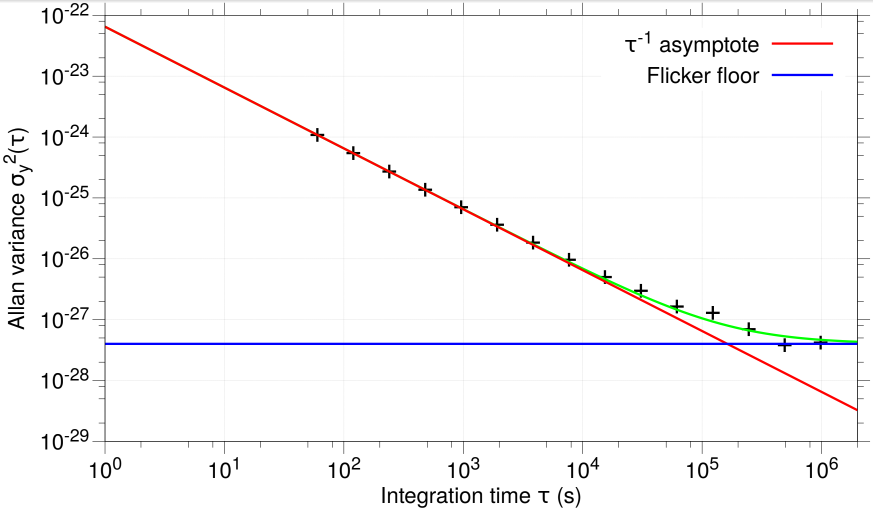

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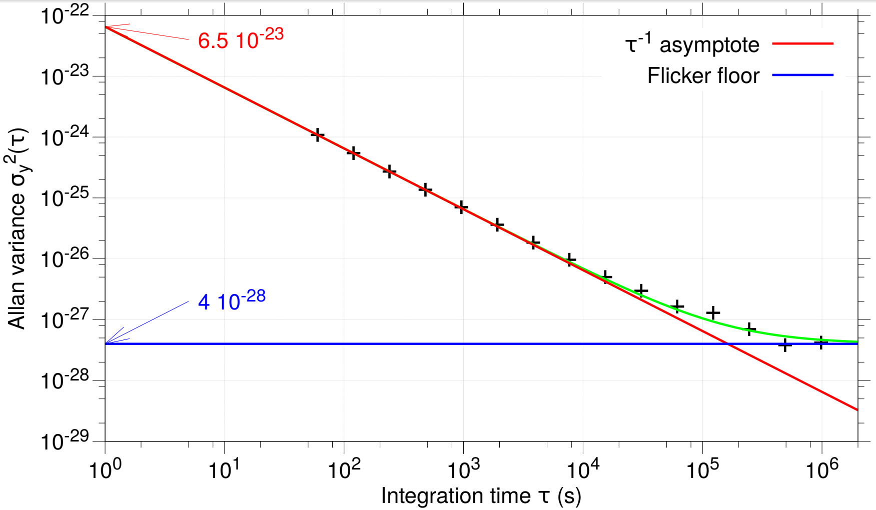

Basic interpretation of an Allan variance curve (II)

Remember. . .

$S_y(f)$	$h_{-2}f^{-2}$	$h_{-1}f^{-1}$	$h_0 f^0$	$h_{+1}f^{+1}$	$h_{+2}f^{+2}$
$\sigma_y^2(\tau)$	$\frac{2\pi^2 h_{-2}\tau}{3}$	$2\ln(2)h_{-1}$	$\frac{h_0}{2\tau}$	$\frac{[1.04 + 3\ln(2\pi f_h \tau)]\, h_{+1}}{4\pi^2\tau^2}$	$\frac{3h_{+2}f_h}{4\pi^2\tau^2}$

- τ^{-1} asymptote: $\Rightarrow$ white FM noise
- τ^0 asymptote: $\Rightarrow$ flicker FM noise (f^{-1} FM)

Basic interpretation of an Allan variance curve (II)

Remember...

$S_y(f)$	$h_{-2}f^{-2}$	$h_{-1}f^{-1}$	$h_0 f^0$	$h_{+1}f^{+1}$	$h_{+2}f^{+2}$
$\sigma_y^2(\tau)$	$\frac{2\pi^2 h_{-2}\tau}{3}$	$2\ln(2)h_{-1}$	$\frac{h_0}{2\tau}$	$\frac{[1.04+3\ln(2\pi f_h \tau)]\,h_{+1}}{4\pi^2\tau^2}$	$\frac{3h_{+2}f_h}{4\pi^2\tau^2}$

- τ^{-1} asymptote: $\Rightarrow$ white FM noise
- τ^{0} asymptote: $\Rightarrow$ flicker FM noise (f^{-1} FM)

Basic interpretation of an Allan variance curve (II)

Remember...

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Basic interpretation of an Allan variance curve (III)

Spectral analysis

- τ^{-1} asymptote: $\Rightarrow$ white FM noise

  $$6.5 \cdot 10^{-23}\tau^{-1} = \frac{h_0}{2\tau} \quad \Rightarrow \quad h_0 = 1.3 \cdot 10^{-22}\ \text{s}$$

- τ^{0} asymptote: $\Rightarrow$ flicker FM noise (f^{-1} FM)

  $$4 \cdot 10^{-28} = 2\ln(2) h_{-1} \quad \Rightarrow \quad h_{-1} = \frac{4 \cdot 10^{-28}}{2\ln(2)} = 2.9 \cdot 10^{-28}$$

$$\Rightarrow S_y(f) = 1.3 \cdot 10^{-22} + 2.9 \cdot 10^{-28} f^{-1}\ \text{s}$$

Time stability estimation

$$\sigma_y(\tau) = \sqrt{6.5 \cdot 10^{-23}\tau^{-1} + 4 \cdot 10^{-28}}$$

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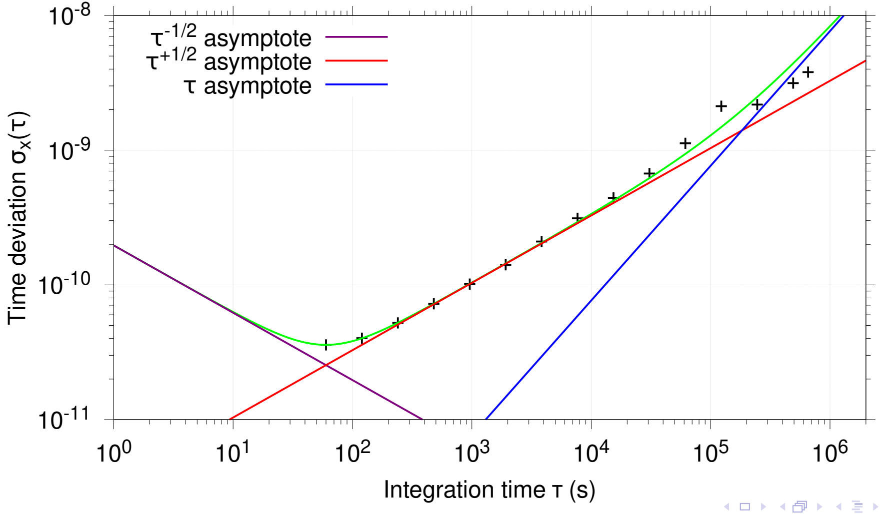
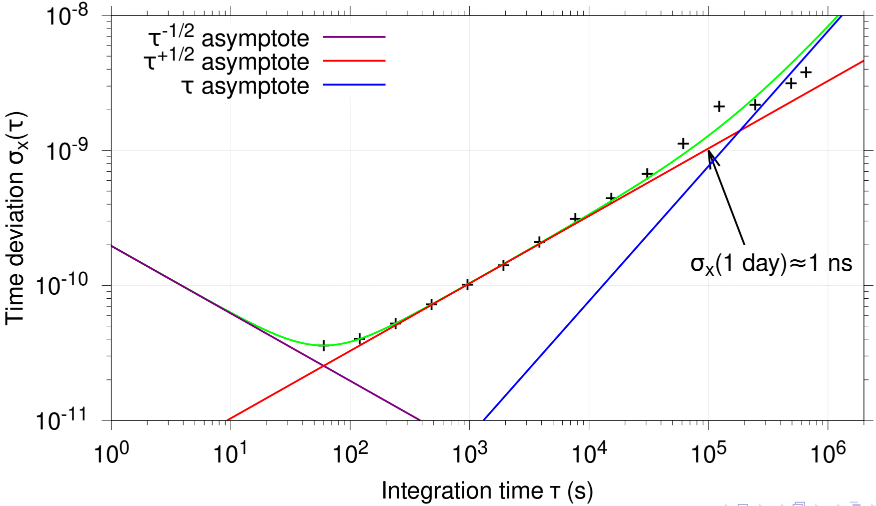
Basic interpretation of a TDev curve



Basic interpretation of a TDev curve



Softwares for frequency stability estimation

- **For windows:** *Stable 32* (W. Riley, M. Danielson, V. Dwivedi)
  Graphical interface, proprietary code, no longer maintained (currently)
  `http://www.stable32.com/`, `https://github.com/IEEE-UFFC/stable32`
- **For unix and macOS:** *SigmaTheta* (F. Vernotte, F. Meyer, A. Kinali, B. Dubois, J.M. Friedt, C. Plantard, P.Y. Bourgeois)
  Collection of many chainable C-modules to build scripts, open source
  `https://gitlab.com/sigmatheta1/`
- **For python (cross-platform):** *AllanTools* (A. Wallin, D. Price, C. Carson, F. Meynadier, Y. Xie, E. Benkler)
  Many python functions, open source
  `https://pypi.org/project/AllanTools/`, `https://github.com/aewallin/allantools`

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