Distribution Sécurisée du Temps et Systèmes Spatiaux

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CONCEPTS ET OUTILS DE LA MÉTROLOGIE TEMPS-FRÉQUENCE

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UNIVERSITE FRANCHE-COMTE







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- 3 Long-term stability: the variances

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 Long-term stability: the variances
 Measurement in the time

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Outline



Time and frequency quantities

- Notations in the time domain
- Frequency noise vs Phase noise
- Measurement in the time domain

2) Frequency domain



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Introduction

Notations in the time domain



$$V(t) = V_0 \sin \left[2\pi\nu_0 t + \varphi\right]$$

where arphi(t) is the phase "*noise*"

• Time error (or phase time) *x*(*t*):

$$V(t) = V_0 \sin \left[2\pi\nu_0 \left(t + x(t)\right)\right]$$
with $x(t) = \frac{\varphi(t)}{2\pi}$ [s]

"My watch is 39 seconds late":

t_{watch} = 10 h 10 min 37 s

• t_{ref} = 10 h,11,11,16, 16,2, , , ,

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Introduction

Notations in the time domain



"My watch is 39 seconds late":

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x(t) = -39 s

"My watch is 39 seconds late":



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Notations in the time domain Frequency noise vs Phase noise Measurement in the time domain

Phase and frequency noise

$$V(t) = V_0 sin [2\pi
u_0 t + arphi(t)]$$

• Instantaneous frequency $\nu(t)$:

$$V(t) = V_0 \sin \left[2\pi\nu(t)t\right]$$
with $\nu(t) = \frac{1}{2\pi} \frac{d\left[2\pi\nu_0 t + \varphi(t)\right]}{dt} = \nu_0 + \frac{1}{2\pi} \frac{d\varphi(t)}{dt}$ [Hz]

• Frequency noise $\Delta \nu(t)$:

$$\Delta\nu(t) = \frac{1}{2\pi} \frac{d\varphi(t)}{dt} \qquad [Hz]$$

• Frequency deviation y(t):

$$y(t) = rac{\Delta
u(t)}{
u_0} = rac{1}{2\pi
u_0} rac{d arphi(t)}{dt} \qquad [dimensionless]$$

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Frequency noise vs Phase noise

Phase and frequency noise: 2 representations of 1 phenomenon

$$\begin{array}{l} x(t) = \frac{\varphi(t)}{2\pi\nu_0} \\ y(t) = \frac{1}{2\pi\nu_0} \frac{d\varphi(t)}{dt} \end{array} \right\} \quad \Rightarrow \quad y(t) = \frac{dx(t)}{dt}$$

A fundamental difference:

- $\varphi(t)$ and x(t) are instantaneous
- $\Delta \nu(t)$ and y(t) have to be averaged Example of a Π -counter:

$$\bar{y}_k = \frac{1}{\tau} \int_{t_k}^{t_k + \tau} y(t) dt = \frac{x(t_k + \tau) - x(t_k)}{\tau}$$

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Notations in the time domain Frequency noise *vs* Phase noise Measurement in the time domain

Phase-time measurements: the time interval counter

Measure the time between

- the rising edge of the DUT signal
- the rising edge of the reference signal

Conversion to frequency deviation: $\bar{y}_k = \frac{x(u_k + \tau) - x}{\tau}$

- Available sampling rate: several MSample/s
- Needs: τ ~1 s!
- \Rightarrow Possible to 'shape' the \bar{y}_k by weighted average:



Other measurements and methods

Frequency counter, phasemeter, beat-note method, frequency comb (optical domain)...

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Phase-time measurements: the time interval counter



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 Noise model

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Time and frequency quantities

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- Short-term stability: the phase noise
- Noise model



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Notations in the frequency domain

Power Spectral Densities (PSD)

• Fourier Transform (finite energy):

$$\Phi(f) = \int_{-\infty}^{+\infty} \varphi(t) e^{-j2\pi f t} \mathrm{d}t \qquad [s]$$

• Energy Spectral Density (finite energy):

$$\left|\Phi(f)\right|^{2} = \left|\int_{-\infty}^{+\infty} \varphi(t) e^{-j2\pi \hbar t} \mathrm{d}t\right|^{2} \qquad [s^{2}]$$

• Power Spectral Density (finite power):

$$arphi(t)= rac{1}{T} \left[arphi(t)e^{-j2\pi t} \mathrm{d}t
ight]^2 \left[s
ight] \equiv [t]$$

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Notations in the frequency domain

Power Spectral Densities (PSD)

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• Energy Spectral Density (finite energy):

$$|\Phi(f)|^2 = \left|\int_{-\infty}^{+\infty} \varphi(t) e^{-j2\pi f t} dt\right|^2 [s^2]$$

• Power Spectral Density (finite power):

$$(f) = \overline{T} \qquad \varphi(t) e^{-j2\pi t} \mathrm{d}t$$

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Notations in the frequency domain

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• Power Spectral Density (finite power):

$$S_{arphi}(t) = \left| \int_{-T/2}^{+T/2} \varphi(t) e^{-j2\pi t t} \mathrm{d} t
ight|^2$$

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$$[s] \equiv [Hz^{-1}]$$

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Notations in the frequency domain

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• Power Spectral Density (finite power):

$$S_{\varphi}(t) = \left| \lim_{t \to \infty} \left\langle \frac{1}{T} \left| \int_{-T/2}^{+T/2} \varphi(t) e^{-j2\pi t t} \mathrm{d}t \right|^2 \right\rangle$$
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Notations in the frequency domain Short-term stability: the phase noise Noise model

Short-term stability: the phase noise

The one-sided PSD

 $\varphi(t)$ is real

$$\mathcal{I}\left[arphi(t)
ight]=0 \quad \Leftrightarrow \quad \mathcal{S}_{arphi}(f)=\mathcal{S}_{arphi}(-f)$$

Definition of the "one-sided" phase Power Spectral density $S_{\varphi}(f)^{1S}$:

$$\left\{ \begin{array}{ll} S_{\varphi}(f)^{1S}=2S_{\varphi}(f)^{2S} & \text{ if } f\geq 0\\ S_{\varphi}(f)^{1S}=0 & \text{ if } f<0 \end{array} \right.$$

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Units on log-log plots

10 log₁₀ [$S_{\varphi}(f)$] is expressed in **dB/Hz**

L(f), a survival from the past

In many data shets, one finds

 $L(f) = 10 \log_{10} \left| \frac{1}{2} S_{\varphi}(f) \right|$ is expressed in **dBc/Hz**

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Short-term stability: the phase noise

Short-term stability: the phase noise

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Notations in the frequency domain Short-term stability: the phase noise Noise model

Relationships between PSD

Time error PSD: $S_x(f)$

•
$$x(t) = \frac{\varphi(t)}{2\pi\nu_0}$$

• Dimension:
$$[s^3] \equiv [Hz^{-3}]$$

Frequency deviation PSD:
$$S_y(f)$$

•
$$y(t) = \frac{1}{2\pi\nu_0} \frac{d\varphi(t)}{dt}$$

$$S_y(f)=rac{f^2}{
u_0^2}S_arphi(f)$$

 $S_y(f) = 4\pi^2 f^2 S_x(f)$

• Dimension: $[s] \equiv [Hz^{-1}]$

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 $S_x(f)=rac{1}{4\pi^2
u_0^2}S_arphi(f)$

Notations in the frequency domain Short-term stability: the phase noise Noise model

Relationships between PSD

Time error PSD: $S_x(f)$

•
$$x(t) = rac{\varphi(t)}{2\pi\nu_0}$$

$$\Rightarrow$$

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$$S_x(f)=\frac{1}{4\pi^2\nu_0^2}S_\varphi(f)$$

• Dimension: $[s^3] \equiv [Hz^{-3}]$

Frequency deviation PSD: $S_y(f)$

•
$$y(t) = \frac{1}{2\pi\nu_0} \frac{d\varphi(t)}{dt} \Rightarrow$$

•
$$y(t) = \frac{dx(t)}{dt} \Rightarrow$$

$$S_{\mathcal{Y}}(f) = \frac{f^2}{\nu_0^2} S_{\varphi}(f)$$

.....

$$S_y(f) = 4\pi^2 f^2 S_x(f)$$

• Dimension: $[s] \equiv [Hz^{-1}]$

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Noise model

The power law noise model

$$S_{y}(f) = \sum_{lpha=-2}^{+2} h_{lpha} f^{lpha} \qquad lpha ext{ integer}$$

$S_y(f)$	$oldsymbol{S}_arphi(oldsymbol{f})$	Noise type	Origin
$h_{-2}f^{-2}$	$b_{-4}f^{-4}$	Random Walk Freq. Mod.	Environment
$h_{-1}f^{-1}$	$b_{-3}f^{-3}$	Flicker F.M.	Resonator
h_0	$b_{-2}f^{-2}$	White F.M.	Thermal noise
h ₁ f	$b_{-1}f^{-1}$	Flicker Phase Mod.	Electronic noise
$h_2 f^2$	b_0	White P.M.	External white noise

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White FM

Random

Walk FM

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White FM vs Random Walk FM





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Phase noise measurement



Courtesy of Ulrich L. Rohde, Synergy Microwave Corporation 99

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Time and frequency quantities The Allan variance (AVAR) Frequency domain Other variances Long-term stability: the variances Practical use of the Allan variance

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Outline

Time and frequency quantities

2) Frequency domain



Long-term stability: the variances

- The Allan variance (AVAR)
- Other variances
- Practical use of the Allan variance

The Allan variance (AVAR) Other variances Practical use of the Allan variance

A statistical estimator

as well as a spectral analysis tool

- Definition of the true variance: $l^{2}(\tau) = \left\langle \left(\bar{y}_{k} - \langle \bar{y}_{k} \rangle \right)^{2} \right\rangle$
- Estimation of the true variance:

$$\sigma^2(\boldsymbol{N},\tau) = \frac{1}{\boldsymbol{N}-1} \sum_{i=1}^{N} \left(\bar{y}_i - \frac{1}{N} \sum_{j=1}^{N} \bar{y}_j \right)^2$$

• The Allan variance (2-sample variance): $\sigma_y^2(\tau) = \sigma^2(2,\tau) = \sum_{i=1}^2 \left(\bar{y}_i - \frac{1}{2} \sum_{j=1}^2 \bar{y}_j \right)^2$



- $\langle
 angle$ stands for:
- ensemble average
- time average

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The Allan variance (AVAR) Other variances Practical use of the Allan variance

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$$\sigma_{y}^{2}(\tau) = \frac{1}{2} \left\langle \left(\bar{y}_{2} - \bar{y}_{1} \right)^{2} \right\rangle = \text{AVAR}(\tau)$$

y_k samples _____

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- ensemble average
- time average
- $\bullet \equiv convolution.$

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- Definition of the true variance: $l^{2}(\tau) = \left\langle \left(\bar{y}_{k} - \langle \bar{y}_{k} \rangle \right)^{2} \right\rangle$
- Estimation of the true variance:

$$\sigma^2(\boldsymbol{N},\tau) = \frac{1}{N-1} \sum_{i=1}^N \left(\bar{y}_i - \frac{1}{N} \sum_{j=1}^N \bar{y}_j \right)^2$$

• The Allan variance (2-sample variance):

$$\sigma_y^2(\tau) = \left\langle \sigma^2(\mathbf{2}, \tau) \right\rangle = \left\langle \sum_{i=1}^2 \left(\bar{y}_i - \frac{1}{2} \sum_{j=1}^2 \bar{y}_j \right)^2 \right\rangle$$

$$\sigma_{\mathbf{y}}^{2}(\tau) = \frac{1}{2} \left\langle \left(\bar{\mathbf{y}}_{2} - \bar{\mathbf{y}}_{1} \right)^{2} \right\rangle = \text{AVAR}(\tau)$$



- $\langle \rangle$ stands for:
- ensemble average
- time average
- \equiv convolution...

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The Allan variance (AVAR) Other variances Practical use of the Allan variance

A spectral analysis tool

as well as a statistical estimator

Convolution in the time domain...

$$\sigma_y^2(\tau) = \left\langle \left[\int_{-\infty}^{+\infty} y(t) h_y(t_k - t) dt \right]^2 \right\rangle$$

with
$$\begin{cases} h_y(t) = \frac{-1}{\sqrt{2}\tau} & \text{if } -\tau \ge t < 0\\ h_y(t) = \frac{+1}{\sqrt{2}\tau} & \text{if } 0 \ge t < \tau\\ h_y(t) = 0 & \text{else} \end{cases}$$

... filtering in the frequency domain

$$\sigma_y^2(\tau) = \int_0^\infty S_y(f) \left| H_y(f) \right|^2 df$$

with $\left| H_y(f) \right|^2 = \left| \text{FT} \left[h_y(t) \right] \right|^2 = 2 \frac{\sin^4(\pi \tau)}{(-\tau)^2}$



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The Allan variance (AVAR) Other variances Practical use of the Allan variance

Link between noise levels and variance responses

$$\sigma_y^2(au) = 2 \int_0^{+\infty} h_lpha f^lpha rac{\sin^4(\pi au f)}{(\pi au f)^2} \mathrm{d}f$$

f_h is the high cut-off frequency



$$\frac{S_{y}(f)}{\sigma_{y}^{2}(\tau)} = \frac{h_{-2}f^{-2}}{3} + \frac{h_{-1}f^{-1}}{2} + \frac{h_{0}f^{0}}{2\tau} + \frac{h_{+1}f^{+1}}{4\pi^{2}\tau^{2}} + \frac{h_{+2}f^{+2}}{4\pi^{2}\tau^{2}} + \frac{h_{+2}f^{+2}}{4\pi^{2}\tau^{2}} + \frac{h_{+2}f_{+2}}{4\pi^{2}\tau^{2}} + \frac$$

and $\sigma_v^2(\tau) = \frac{1}{2}D_1^2\tau^2$ for a linear frequency drift: $y(t) = D_1 t$

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$$\frac{S_{y}(f) \| h_{-2}f^{-2} \| h_{-1}f^{-1} \| h_{0}f^{0} \| h_{+1}f^{+1} \| h_{+2}f^{+2}}{\sigma_{y}^{2}(\tau) \| \frac{2\pi^{2}h_{-2}\tau}{3} \| 2\ln(2)h_{-1} \| \frac{h_{0}}{2\tau} \| \frac{[1.04+3\ln(2\pi f_{h}\tau)]h_{+1}}{4\pi^{2}\tau^{2}} \| \frac{3h_{+2}f_{h}}{4\pi^{2}\tau^{2}}}{and} \sigma_{y}^{2}(\tau) = \frac{1}{2}D_{1}^{2}\tau^{2} \text{ for a linear frequency drift: } y(t) = D_{1}t$$

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The Allan variance (AVAR) Other variances Practical use of the Allan variance

Allan variance with or without overlapping

Allan variance with overlapping



τ_0 -steps moving average

Benefits and drawbacks :

• lower dispersion

more correlated estimates

Allan variance without overlapping



Shifted by τ -steps : $\tau = 3\tau_0 \Leftrightarrow \overline{Y}_1 = (\overline{y}_1 + \overline{y}_2 + \overline{y}_3)/3$

Benefits and drawbacks :

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The Allan variance (AVAR) Other variances Practical use of the Allan variance

Allan variance versus Allan deviation

$$\mathsf{ADEV}(au) = \sigma_y(au) = \sqrt{\sigma_y^2(au)}$$

Physical meaning

• $\sigma_y(\tau) \equiv -\frac{\tau}{\tau}$ *Ex.:* Cs clock $\sigma_y(\tau = 1 \text{ day}) = 10^{-14} \Rightarrow \Delta t \approx 10^{-14} \cdot 10^5 = 10^{-9} = 1 \text{ ns over 1 day}$ • $\sigma_y(\tau) \equiv \frac{\Delta f}{\nu_0}$ (during τ) *Ex.:* H-Maser @ 100 MHz $\sigma_y(\tau = 1\text{ h}) = 10^{-14} \Rightarrow \Delta f \approx 10^{-14} \cdot 10^8 = 10^{-6} = 1 \mu \text{Hz over 1 h}$

Benefits and drawbacks

Easy to interpret

Biased

inition curves, always use the Allan variance.

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The Allan variance (AVAR) Other variances Practical use of the Allan variance

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The Allan variance (AVAR) Other variances Practical use of the Allan variance

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Never fit the Allan deviation curves, always use the Allan variance!

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The Allan variance (AVAR) Other variances Practical use of the Allan variance

The most widely used variances

• The Hadamard variance (Picinbono):

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$$\sigma_{H}^{2}(\tau) = rac{1}{6} \left\langle (-\bar{y}_{1} + 2\bar{y}_{2} - \bar{y}_{3})^{2}
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angle.$$

$$|H_H(f)|^2 = \frac{8}{3} \frac{\sin^6(\pi \tau f)}{(\pi \tau f)^2}.$$

The modified Allan variance (MVAR):

$$\mathsf{Mod}\sigma_{y}^{2}(\tau) = \frac{1}{2} \left\langle \left(\frac{1}{n} \sum_{i=1}^{n} \bar{y}_{i+n} - \bar{y}_{i} \right)^{2} \right\rangle_{I}$$
$$|H_{\mathcal{M}}(f)|^{2} = \frac{\sin^{6}(\pi \tau f)}{(\pi \tau f)^{2} n^{2} \sin^{2}(\pi \tau_{0} f)}.$$

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• The time variance (TVAR): $\sigma_x^2(\tau) = \frac{\tau^2}{2} \text{Mod}\sigma_y^2(\tau)$.

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The Allan variance (AVAR) Other variances Practical use of the Allan variance

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applications

The Allan variance (AVAR) Other variances Practical use of the Allan variance

Responses of the variances

$S_y(f)$	$\sigma_{H}^{2}(au)$	$Mod\sigma_y^2(au)$	$\sigma_x^2(\tau)$
$(s\equiv Hz^{-1})$			
$h_{-4}f^{-4}$	$\frac{44\pi^4 h_{-4}\tau^3}{60}$	-	-
$h_{-3}f^{-3}$	$\frac{[27\ln(2)-32\ln(3)]\pi^2h_{-3}\tau^2}{6}$	-	-
$h_{-2}f^{-2}$	$\frac{\pi^2 h_{-2}\tau}{3}$	$\frac{11\pi^2 h_{-2}\tau}{20}$	$\frac{11\pi^2 h_{-2}\tau^3}{60}$
$h_{-1}f^{-1}$	$\frac{[8\ln(2)-3\ln(3)]h_{-1}}{2}$	$\frac{[27\ln(3) - 32\ln(2)]h_{-1}}{8}$	$\frac{[27\ln(3) - 32\ln(2)]h_{-1}}{24\tau^2}$
$h_0 f^0$	$\frac{h_0}{2\tau}$	$\frac{h_0}{4\tau}$	$\frac{h_0\tau}{12}$
$h_{+1}f^{+1}$	$\frac{5[0,964+\ln(\pi\tau f_h)]h_{+1}}{6\pi^2\tau^2}$	$\frac{[24\ln(2)-9\ln(3)]h_{+1}}{8\pi^2\tau^2}$	$\frac{[8\ln(2) - 3\ln(3)]h_{+1}}{8\pi^2}$
$h_{+2}f^{+2}$	$\frac{5h_{+2}f_h}{6\pi^2\tau^2}$	$\frac{3h_{+2}}{8\pi^2\tau^3}$	$rac{h_{+2}}{8\pi^2 au}$

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Time Interval Error



TIE

The Time Interval Error (TIE) is the difference between the extrapolation of the clock model and the clock state at a given time.

The Maximum Time Interval Error (MTIE) can be computed from **TDev** and the type of noise.

Vernotte, J. Delporte, M. Brunet, and T. Tournier. Uncertainties

f drift coefficients and extrapolation errors: Application to clock

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F. Vernotte, J. Delporte, M. Brunet, and T. Tournier. Uncertainties of drift coefficients and extrapolation errors: Application to clock error prediction. Metrologia. 38(4):325–342. 2001.

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Practical use of the Allan variance Long-term stability: the variances Basic interpretation of an Allan variance curve (I) 10-22-10⁻²³-10⁻²⁴-+Allan variance $\sigma_y{}^2(\tau)$ 10-25 10⁻²⁶-10⁻²⁷ 10⁻²⁸-10⁻²⁹-106 100 10^{2} 10³ 105 10¹ 10⁴ Integration time τ (s)

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Basic interpretation of an Allan variance curve (II)

Remember...

$$\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline S_y(f) & h_{-2}f^{-2} & h_{-1}f^{-1} & h_0f^0 & h_{+1}f^{+1} & h_{+2}f^{+2} \\ \hline \sigma_y^2(\tau) & \frac{2\pi^2h_{-2}\tau}{3} & 2\ln(2)h_{-1} & \frac{h_0}{2\tau} & \frac{\left[1.04+3\ln(2\pi f_h\tau)\right]h_{+1}}{4\pi^2\tau^2} & \frac{3h_{+2}f_h}{4\pi^2\tau^2} \\ \hline \end{array}$$

• τ^{-1} asymptote: \Rightarrow white FM noise • τ^{0} asymptote: \Rightarrow flicker FM noise (f^{-1} F

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Basic interpretation of an Allan variance curve (II)

Remember...

• τ^{-1} asymptote: \Rightarrow white FM noise

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The Allan variance (AVAR) Other variances Practical use of the Allan variance

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The Allan variance (AVAR) Other variances Practical use of the Allan variance

Basic interpretation of an Allan variance curve (III)

Spectral analysis

• τ^{-1} asymptote: \Rightarrow white FM noise $6.5 \cdot 10^{-23} \tau^{-1} = \frac{h_0}{2\tau} \Rightarrow h_0 = 1.3 \cdot 10^{-22} \text{ s}$ • $h_0 = 1.3 \cdot 10^{-22} \text{ s}$ 4 $10^{-28} = 2\ln(2)h_0$ $\Rightarrow h_0 = 2\ln(2)$ $S_1(t) = 1.3 \cdot 10^{-22} + 2.9 \cdot 10^{-28} t^{-1} \text{ s}$

Time stability estimation

$$\sigma_y(\tau) = \sqrt{6.5 \cdot 10^{-23} \tau^{-1} + 4 \cdot 10^{-28}}$$

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Spectral analysis

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Time stability estimation

$$\sigma_{\gamma}(\tau) = \sqrt{6.5 \cdot 10^{-23} \tau^{-1} + 4 \cdot 10^{-28}}$$

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The Allan variance (AVAR) Other variances Practical use of the Allan variance

Basic interpretation of an Allan variance curve (III)

Spectral analysis

•
$$\tau^{-1}$$
 asymptote: \Rightarrow white FM noise
 $6.5 \cdot 10^{-23} \tau^{-1} = \frac{h_0}{2\tau} \Rightarrow h_0 = 1.3 \cdot 10^{-22} \text{ s}$
• τ^0 asymptote: \Rightarrow flicker FM noise $(f^{-1} \text{ FM})$
 $4 \cdot 10^{-28} = 2 \ln(2) h_{-1} \Rightarrow h_{-1} = \frac{4 \cdot 10^{-28}}{2 \ln(2)} = 2.9 \cdot 10^{-28}$
 $\Rightarrow S_y(f) = 1.3 \cdot 10^{-22} + 2.9 \cdot 10^{-28} f^{-1} \text{ s}$

Time stability estimation

$\sigma_{\gamma}(\tau) = \sqrt{6.5 \cdot 10^{-23} \tau^{-1} + 4 \cdot 10^{-28}}$

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Spectral analysis

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 $\Rightarrow S_y(f) = 1.3 \cdot 10^{-22} + 2.9 \cdot 10^{-28} f^{-1} \text{ s}$

Time stability estimation

$$\sigma_{\rm y}(\tau) = \sqrt{6.5 \cdot 10^{-23} \tau^{-1} + 4 \cdot 10^{-28}}$$

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• $\tau < 1$ day: $\sigma_y(\tau) \approx 8 \cdot 10^{-12} \tau^{-1/2}$

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Basic interpretation of an Allan variance curve (III)

Spectral analysis

•
$$\tau^{-1}$$
 asymptote: \Rightarrow white FM noise
 $6.5 \cdot 10^{-23}\tau^{-1} = \frac{h_0}{2\tau} \Rightarrow h_0 = 1.3 \cdot 10^{-22} \text{ s}$
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 $4 \cdot 10^{-28} = 2 \ln(2)h_{-1} \Rightarrow h_{-1} = \frac{4 \cdot 10^{-28}}{2 \ln(2)} = 2.9 \cdot 10^{-28}$
 $\Rightarrow S_y(f) = 1.3 \cdot 10^{-22} + 2.9 \cdot 10^{-28} f^{-1} \text{ s}$

Time stability estimation

$$\sigma_y(au) = \sqrt{6.5\cdot 10^{-23} au^{-1} + 4\cdot 10^{-28}}$$

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•
$$\tau < 1$$
 day: $\sigma_y(\tau) \approx 8 \cdot 10^{-12} \tau^{-1/2}$

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The Allan variance (AVAR) Other variances Practical use of the Allan variance

Basic interpretation of an Allan variance curve (III)

Spectral analysis

•
$$\tau^{-1}$$
 asymptote: \Rightarrow white FM noise
 $6.5 \cdot 10^{-23}\tau^{-1} = \frac{h_0}{2\tau} \Rightarrow h_0 = 1.3 \cdot 10^{-22} \text{ s}$
• τ^0 asymptote: \Rightarrow flicker FM noise $(f^{-1} \text{ FM})$
 $4 \cdot 10^{-28} = 2 \ln(2)h_{-1} \Rightarrow h_{-1} = \frac{4 \cdot 10^{-28}}{2 \ln(2)} = 2.9 \cdot 10^{-28}$
 $\Rightarrow S_y(f) = 1.3 \cdot 10^{-22} + 2.9 \cdot 10^{-28} f^{-1} \text{ s}$

Time stability estimation

$$\sigma_y(au) = \sqrt{6.5\cdot 10^{-23} au^{-1} + 4\cdot 10^{-28}}$$

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•
$$\tau < 1$$
 day: $\sigma_y(\tau) \approx 8 \cdot 10^{-12} \tau^{-1/2}$
• $\tau > 1$ day: $\sigma_y(\tau) \approx 2 \cdot 10^{-14}$

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Long-term stability: the variances

Practical use of the Allan variance

Basic interpretation of a TDev curve



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Long-term stability: the variances

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Softwares for frequency stability estimation

- For windows: *Stable 32* (W. Riley, M. Danielson, V. Dwivedi) Graphical interface, proprietary code, no longer maintained (currently) http://www.stable32.com/, https://github.com/IEEE-UFFC/stable32
- For unix and macOS: SigmaTheta (F. Vernotte, F. Meyer, A. Kinali, B. Dubois, J.M. Friedt, C. Plantard, P.Y. Bourgeois) Collection of many chainable C-modules to build scripts, open source https://gitlab.com/sigmathetal/
- For python (cross-platform): AllanTools (A. Wallin, D. Price, C. Carson, F. Meynadier, Y. Xie, E. Benkler)
 Many python functions, open source

https://pypi.org/project/AllanTools/, https://github.com/aewallin/allantools

Time and frequency quantities Frequency domain Long-term stability: the variances The Allan variance (AVAR) Other variances Practical use of the Allan variance

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